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FREE OSCILLATIONS OF A NON-LINEAR CUBIC SYSTEM WITH TWO DEGREES OF FREEDOM AND CLOSE NATURAL FREQUENCIES†

YE. V. LADYGINA and A. I. MANEVICH

Dnepropetrovsk

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The multiscale method [1] is used to investigate free oscillations of a conservative system with two degrees of freedom having cubic non-linearities (of symmetric nature) and close natural frequencies. Possible oscillation modulation regimes are found which depend on the coefficients of the system of differential equations, the energy and initial conditions.

FOR SYSTEMS with two degrees of freedom which have quadratic non-linearities, an internal resonance at the frequency ratio 1:2 has been studied, along with an internal resonance for systems with cubic non-linearities and the frequency ratio 1:3 [1]. In recent years attention has turned to mode interactions (of internal resonance type) for close natural frequencies. Experimental observations and solutions of particular problems show that this effect is relevant to the description of oscillatory processes in suspension bridges [2–4], cylindrical shells and other constructions [5–7]. However, the literature does not contain any general analysis of mode interactions of free oscillations in non-linear systems with close natural frequencies. In particular, we do not know what types of oscillation modulation are possible, what determines the degree and period of energy transfer in a system, what is the number of steady-state regimes (without modulation), which of them are stable, etc.

1. AMPLITUDE-FREQUENCY MODULATION EQUATIONS

Consider a non-linear oscillatory system (initially, for generality, with damping), described by the equations

$$\ddot{u}_k + 2\mu_* \dot{u}_k + \omega_*^2 u_k = b_{kk} u_k^3 + b_{12} u_1^k u_2^{3-k}, \qquad k = 1,2 \tag{1.1}$$

The frequencies ω_1 and ω_2 are assumed to be close, and the damping factors for the two modes are taken to be the same.

Equations (1.1) give the general case of systems with symmetric potentials (when $\mu_* = 0$) that include terms of the second and fourth degree. They are similar to the Duffing equation for systems with one degree of freedom and describe a broad class of mechanical systems. (For generality, no restrictions are imposed on the coefficients b_{ij}.)

In accordance with the multiscale method we introduce "fast" and "slow" times $T_0 = t$, $T_1 = \varepsilon T_0$, $T_2 = \varepsilon^2 T_0$. (The time T_1 will not be necessary below.) We will seek a solution of system (1.1) in the form of an expansion

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$$u_k = \varepsilon u_{k1}(T_0, T_1, T_2, \ldots) + \varepsilon^3 u_{k3}(T_0, T_1, T_2, \ldots) + \ldots \tag{1.2}$$

(terms of order ε^2 vanish for a system with cubic non-linearities).

The smallness of μ_* and the frequency difference are introduced through the conditions

$$\mu_* = \varepsilon^2 \mu, \qquad \omega_1 = \omega, \qquad \omega_2^2 = \omega^2 + \varepsilon^2 \sigma \tag{1.3}$$

Using

$$d/dt = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots (D_0 = \partial/\partial T_0, D_1 = \partial/\partial T_1, D_2 = \partial/\partial T_2),$$
$$d^2/dt^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (2D_0 D_2 + D_1^2) + \ldots$$

we obtain the following systems of equations for the two approximations

$$D_0^2 u_{k1} + \omega^2 u_{k1} = 0 \tag{1.4}$$

$$D_0^2 u_{k3} + \omega^2 u_{k3} = -2D_0(D_2 u_{k1} + \mu u_{k1}) + b_{kk} u_{k1}^3 + b_{12} u_{11}^k u_{21}^{3-k} - \delta_{2k} \sigma u_{k1} \tag{1.5}$$

(where δ_{ij} is the Kronecker delta).

We will write the solution of system (1.4) in the form

$$u_{k1} = A_k(T_2)\exp(i\omega T_0) + \bar{A}_k(T_2)\exp(-i\omega T_0) \tag{1.6}$$

(where the bars denote complex conjugation).

Substituting (1.6) into system (1.5), from the condition for there to be no secular terms in the resulting equations we have

$$-2i\omega(A_k' + \mu A_k) + 3b_{kk} A_k^2 \bar{A}_k + 2b_{12} A_k A_{3-k} \bar{A}_{3-k} + b_{12} A_k A_{3-k}^2 - \delta_{2k} \sigma A_k = 0 \tag{1.7}$$

(where the prime denotes differentiation with respect to T_2).

Putting the complex amplitude in exponential form $A_k = \frac{1}{2} a_k \exp(i\theta_k)$ $(k = 1, 2)$ we separate (1.7) into real and imaginary parts and obtain a system of equations governing the amplitude modulation and phase of both modes

$$(a_k^2)' + 2\mu a_k^2 = (-1)^k b_{12} (4\omega)^{-1} a_1^2 a_2^2 \sin 2\gamma \tag{1.8}$$

$$8\omega\theta_k' = -3b_{kk} a_k^2 - b_{12} a_{3-k}^2 (2 + \cos 2\gamma) \tag{1.9}$$

Having eliminated sin 2γ we obtain from Eqs (1.8) the integral

$$a_1^2 + a_2^2 = E \cdot \exp(-2\mu T_2) = E \cdot \exp(-2\varepsilon^2 \mu t) \tag{1.10}$$

where the arbitrary constant E is proportional to the energy of the system (to a first approximation). In particular, for a conservative system ($\mu = 0$)

$$a_1^2 + a_2^2 = E \tag{1.11}$$

From (1.9) we obtain the equation for the phase difference γ

$$8\omega\gamma' = (3b_{11} - 2b_{12})a_1^2 + (2b_{12} - 3b_{22})a_2^2 + b_{12}(a_2^2 - a_1^2)\cos 2\gamma + 4\sigma \tag{1.12}$$

For further analysis it is convenient [1] to change to the new variable $\xi = a_1^2/E$ $(0 \leqslant \xi \leqslant 1)$.

Then from Eqs (1.9) with $k=1$ and (1.12) we obtain a system of equations governing the amplitude-frequency modulation in ξ, γ variables

$$\begin{aligned} \xi' &= -2\mu\xi + \Gamma_0\xi(1-\xi)\sin 2\gamma \\ \gamma' &= \xi\Gamma_1 + \Gamma_2 + \Gamma_0(\tfrac{1}{2}-\xi)\cos 2\gamma \end{aligned} \tag{1.13}$$

Here

$$\begin{aligned} \Gamma_0 &= b_{12}E(4\omega)^{-1}, \qquad \Gamma_1 = (3b_{11} - 4b_{12} + 3b_{22})E(8\omega)^{-1} \\ \Gamma_2 &= [(2b_{12} - 3b_{22})E + 4\sigma](8\omega)^{-1} \end{aligned} \tag{1.14}$$

Without loss of generality we shall take $\Gamma_0 \neq 0$, because otherwise $b_{12}=0$ and system (1.1) decompose into two decoupled equations.

2. SOLUTION OF THE MODULATION EQUATIONS

We will perform further analysis for the case of a conservative system ($\mu=0$). Dividing the second equation of (1.13) by the first, we obtain

$$\frac{d\gamma}{d\xi} = \frac{\xi(\Gamma_1 - \Gamma_0\cos 2\gamma) + \Gamma_2 + \tfrac{1}{2}\Gamma_0\cos 2\gamma}{\Gamma_0\xi(1-\xi)\sin 2\gamma} \tag{2.1}$$

The solution of this ordinary differential equation is

$$\Gamma_0\xi(1-\xi)\cos 2\gamma + \Gamma_1\xi^2 + 2\Gamma_2\xi = C \tag{2.2}$$

where C, the constant of integration, determines the trajectory in the (ξ, γ) plane—the "amplitude-phase portrait" of the system (the AP-portrait). Eliminating γ from the first equation of (1.13) and (2.2), we obtain

$$\begin{aligned} &\Gamma_0^{-2}(d\xi/dT_2)^2 = F_1^2(\xi) - F_2^2(\xi) \\ &F_1(\xi) = \xi(1-\xi), \qquad F_2(\xi) = \Gamma_0^{-1}(\Gamma_1\xi^2 + 2\Gamma_2\xi - C) \end{aligned} \tag{2.3}$$

The form of this equation is identical with that derived in [1] for the case when $\omega_2 = 3\omega_1$, but the functions $F_1(\xi)$ and $F_2(\xi)$ are of different form.

The condition for the solution of Eq. (2.3) to exist

$$|F_1| \geqslant |F_2| \tag{2.4}$$

means that the solutions correspond to parabolic segments $F_2(\xi)$ inside the domain bounded by the parabolic arcs $\pm F_1(\xi)$ over the interval [0, 1] (Fig. 1). The points of intersection of the $F_2(\xi)$ parabolas with the ξ axis, as can be seen from (2.2), correspond to the condition $\cos 2\gamma = 0$, i.e. $\gamma = \pm(2n+1)\pi/4$ ($n = 0, 1, 2, \ldots$), or the values $\xi = 0$ and $\xi = 1$. The points of intersection of the parabolas $F_2(\xi)$ and $F_1(\xi)$ correspond to extremal values of the functions $\xi(T_2)$ and $a_k(T_2)$, respectively ($k = 1, 2$). It follows from Eqs (1.13) (for $\mu = 0$) that at these points $\sin 2\gamma = 0$ (if $\xi \neq 0$ and $\xi \neq 1$), i.e. $\gamma = \pm n\pi/2$ ($n = 0, 1, 2, \ldots$), and that the points on the lower curve ($F_1 < 0$) correspond to even values of n and those on the upper curve to odd values. Consequently, the minimum and maximum values of ξ governing the amplitude modulations and degree of energy transfer between the modes are equal to the roots of the equations $F_1(\xi)=\pm F_2(\xi)$, i.e. the equations

$$\xi^2(\mp\Gamma_1-\Gamma_0)+\xi(\Gamma_0\mp 2\Gamma_2)\pm C=0 \tag{2.5}$$

where the upper and lower signs correspond to points of intersection of the parabola $F_2(\xi)$ with the upper and lower curves $\pm F_1(\xi)$, respectively, while the value of C is governed by the initial values ξ_0 and γ_0.

The solutions of Eq. (2.5) and the construction of a "characteristic graph" (Fig. 1) give a graphical representation of the oscillatory regime. There are two basic ways in which the curves $F_1(\xi)$ and $\pm F_2(\xi)$ can intersect in a "coarse" system, corresponding to the two basic oscillatory regimes:

1. both points of intersection lie on the same parabola $+F_1(\xi)$ or $-F_1(\xi)$ (curves 1 and 2 in Fig. 1a);

2. the parabola $F_2(\xi)$ intersects both the parabolas $+F_1(\xi)$ and $-F_1(\xi)$ in the interval [0, 1].

In the first case, the phase difference γ will oscillate about the value $\gamma=\pm n\pi/2$. Synchronization of the oscillations proceeds "on average" over the modulation period: at the times when the extrema a_1 and a_2 are achieved the oscillations of both degrees of freedom proceed either in phase or in antiphase, if both points of intersection lie on the lower branch, or the phase difference at these instants is equal to $\pi/2$, $(3\pi/2)$ if both points lie on the $F_1>0$ branch.

In the second case the phase difference increases monotonically, running through the sequential values $n\pi/2$ ($n=0, 1, \ldots$) at the extremal times. These two types of oscillatory regime with oscillating and monotonically increasing phase differences will respectively be called modulations of the first and second type.

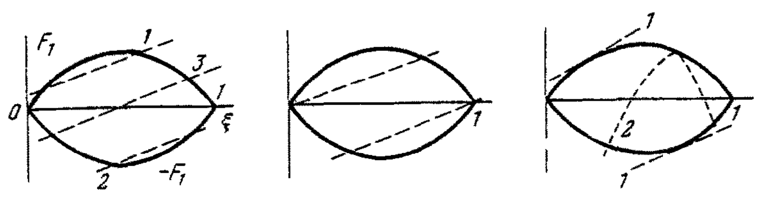
The solution of Eq. (2.3) has the form

$$\pm\frac{1}{|\Gamma_0|}\int_{\xi_0}^{\xi}[F_1^2(\xi)-F_2^2(\xi)]^{-1/2}d\xi=T_2-T_{20},\qquad \xi_0=\xi(T_{20}) \tag{2.6}$$

Suppose $\xi_1,\ldots,\xi_4$ are the roots of the fourth-degree polynomial $F_1^2(\xi)-F_2^2(\xi)$, arranged in increasing order, with ξ_2 and ξ_3 lying inside the domain bounded by the parabolas $\pm F_1(\xi)$. The modulation semiperiod (for the oscillating phase case) corresponds to ξ varying over the interval (ξ_2, ξ_3) and so the modulation period is equal to

$$T^*=\frac{2}{|\Gamma_0||1-\Gamma_1^2|^{1/2}}\int_{\xi_2}^{\xi_3}[(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)]^{-1/2}d\xi \tag{2.7}$$

For "non-coarse" systems one must consider singular cases for the position of the $F_2(\xi)$ curve (Figs 1b and c): the passage of $F_2(\xi)$ through the points $\xi=0$ or $\xi=1$ (Fig. 1b), and "external" or "internal" touching of the parabolas $F_1(\xi)$ and $F_2(\xi)$ (corresponding to lines 1 and 2 in Fig. 1c). In these cases two of the roots ξ_j coincide: in the first case $\xi_1=\xi_2=0$ or



(a) (b) (c)

FIG. 1.

$\xi_3=\xi_4=1$, and in cases 2 and 3 $\xi_2=\xi_3$. Because the improper integral in (2.7) diverges when two of the ξ_j roots coincide, the modulation period tends to infinity as these regimes are approached. These are "boundary" regimes separating modulations of the two types distinguished above (lines 1 and 3) and associated with separatrices in the (ξ, γ) plane, or regimes of stationary oscillations without modulation (curve 2). We remark that the "aperiodic" oscillations described in [4] correspond to these boundary regimes.

3. STATIONARY POINTS, SEPARATRICES AND AMPLITUDE-PHASE PORTRAITS

We will consider possible AP-portraits in the (ξ, γ) plane which are given by integral (2.2) and which graphically describe the oscillatory modes of the system.

Stationary points corresponding to oscillations with no modulation are found using (1.13) from the system of equations

$$\xi(1-\xi)\sin 2\gamma=0 \qquad (3.1)$$
$$\xi\Gamma_1+\Gamma_2+\Gamma_0(\tfrac{1}{2}-\xi)\cos 2\gamma=0$$

which can have the following solutions

$$\xi=0, \qquad \cos 2\gamma=-2\Gamma_2/\Gamma_0 \qquad (3.2)$$

$$\xi=1, \qquad \cos 2\gamma=-2(\Gamma_1+\Gamma_2)/\Gamma_0 \qquad (3.3)$$

$$\gamma=\pm n\pi/2 \qquad (n=0,1,2,\ldots), \quad \xi=\xi_*^{\pm}=(\pm\Gamma_0/2-\Gamma_2)/(\Gamma_1\pm\Gamma_0) \qquad (3.4)$$

These solutions exist when the following conditions are satisfied

$$(1)\ |2\Gamma_2|\leqslant|\Gamma_0|, \quad (2)\ |\Gamma_1+\Gamma_2|\leqslant|\Gamma_0|, \quad (3)\ 0\leqslant\xi_*^{+}\leqslant 1, \quad (4)\ 0\leqslant\xi_*^{-}\leqslant 1 \qquad (3.5)$$

Using the periodicity with respect to γ we will confine ourselves to the plane rectangle $(0\leqslant\xi\leqslant 1,\ 0\leqslant\gamma\leqslant\pi)$. The stationary points (3.2)–(3.4) can be positioned on the boundary lines of this rectangle and on the mid-line $\gamma=\pi/2$ (with not more than one point on a line). It is easiest to investigate the nature of a stationary point with the help of (2.2), considering the form of the integral curves in a neighbourhood of the stationary point. The stationary points at $\xi=0$ and $\xi=1$ are saddle points and therefore unstable. From this it follows that the presence of a second degree of freedom makes oscillations along the first generalized coordinate unstable if the stationary points (3.2) or (3.3) exist.

In the neighbourhoods of stationary points on the lines $\gamma=\pm n\pi/2$ the trajectories can be of either elliptic of hyperbolic type, and consequently, these stationary points can be stable or unstable. The stability conditions for odd and even n, respectively, have the forms

$$(5)\ \Gamma_0(\Gamma_0+\Gamma_1)>0, \qquad (6)\ \Gamma_0(\Gamma_0-\Gamma_1)>0 \qquad (3.6)$$

On the characteristic graph Fig. 1(c) the "externally" touching hyperbolas (curve 1) correspond to stable stationary points and the "internally" touching ones (curve 2) to unstable points.

Stationary points on the $\gamma=\pm n\pi/2$ lines correspond to synchronous single-frequency modes, i.e. normal oscillations of the non-linear system [8]. It follows from (1.2) and (1.6) that points on the lines $\gamma=0$ and $\gamma=\pi$ correspond in the (u_1, u_2) configuration space to the two straight lines $u_2=\pm hu_1$, where

$$h=\frac{a_2}{a_1}=\left(\frac{1-\xi_*^{-}}{\xi_*^{-}}\right)^{1/2}=\left(\frac{\Gamma_0/2-\Gamma_1-\Gamma_2}{\Gamma_0/2+\Gamma_2}\right)^{1/2}$$

Stationary points on the $\gamma=\pi/2$ and $3\pi/2$ lines correspond to the ellipses

$$\frac{u_1^2}{\xi_*^+}+\frac{u_2^2}{1-\xi_*^+}=E\varepsilon^2$$

and points on the $\xi=0$ and $\xi=1$ lines correspond to straight lines along the Ou_2 and Ou_1 axes.

The separatrices pass through the possible unstable stationary points. For separatrices passing through the "left" points (3.2) one should put $C=0$ in (2.2). We obtain equations for two branches

$$(1)\ \xi=0,\qquad (2)\ \cos 2\gamma=-(\xi\Gamma_1+2\Gamma_2)/[\Gamma_0(1-\xi)] \tag{3.7}$$

which exist when condition 1 of (3.5) is satisfied. A "right" separatrix, passing through the stationary points (3.3), exists when condition 2 is satisfied. The equations of the branches of this separatrix are obtained from (2.2) with $C=\Gamma_1+2\Gamma_2$

$$(1)\ \xi=1,\qquad (2)\ \cos 2\gamma=[(\xi+1)\Gamma_1+2\Gamma_2]/(\Gamma_0\xi) \tag{3.8}$$

The central separatrix (CS) passing through the stationary points (4.3) for odd (or even) n exists when condition 3 (condition 4) is satisfied and condition 5 (condition 6) is violated. Substituting the coordinates of point (3.4) into (2.2), we obtain $C=(-\Gamma_2\pm\Gamma_0/2)^2/(\mp\Gamma_0-\Gamma_1)$ and equations for the branches of the central separatrix

$$\xi=B\pm\sqrt{B^2-D}$$
$$B=\frac{\Gamma_0\cos 2\gamma+\Gamma_2}{\Gamma_0\cos 2\gamma-\Gamma_1},\qquad D=\frac{(\Gamma_2\mp\Gamma_0/2)^2}{(\Gamma_1\pm\Gamma_0)(\Gamma_1-\Gamma_0\cos 2\gamma)} \tag{3.9}$$

The stationary points and separatrices possess the following properties.

1. If "left" stationary points (3.2) exist (i.e. condition 1 is satisfied), then in the rectangle $(0\leqslant\xi\leqslant 1,\ 0\leqslant\gamma<\pi)$ there is at least one "intermediate" stationary point (3.4) on the line $\gamma=\pi/2$ or $\gamma=0$, and this point is stable.

Indeed, when condition 1 is satisfied the sign of the numerators in condition 3, 4 is given by the sign of their first term, and for their moduli we have $|\pm\Gamma_0/2-\Gamma_2|\leqslant|\Gamma_0|$, if the signs of Γ_1 and Γ_0 are the same, then the sign of the denominator in condition 3 is the same as the sign of the numerator, and because we then have $|\Gamma_1+\Gamma_0|>|\Gamma_0|$, condition 3 is satisfied and, clearly, condition 5. In the case of unlike signs for Γ_1 and Γ_0, the signs of the numerator and denominator in condition 4 are the same and $|\Gamma_1-\Gamma_0|>|\Gamma_0|$, so that conditions 4 and 6 are satisfied.

A similar assertion holds for the "right" stationary points (3.3).

2. If one stationary unstable point (3.4) exists on the line $\gamma=\pi/2$ (or $\gamma=0$), then a stable stationary point exists on the line $\gamma=0$ (or $\gamma=\pi/2$); here there are no separatrices (3.7) and (3.8).

Suppose condition 3 is satisfied and condition 5 is not satisfied (i.e. the stationary point at $\gamma=\pi/2$ is unstable). Then Γ_0 and Γ_1 have unlike signs and $|\Gamma_1|>|\Gamma_0|$. It follows from condition 3 (because the sign of the denominator is governed by the sign of Γ_1 and is opposite to the sign of Γ_0) that the signs of Γ_0 and Γ_2 are the same and $|\Gamma_2|>|\Gamma_1|/2$. Condition 1 is therefore violated. Considering the cases $\Gamma_1>0$ and $\Gamma_1<0$ separately, and taking into account that the sign of Γ_1 is opposite to the signs of Γ_0 and Γ_2 and that $|\Gamma_1|>|\Gamma_0|$, we find that in both cases condition 2 is violated, and the (right) inequality in condition 4 is also violated, which proves the assertion.

These properties enables us to describe the various possible AP-portraits in the (ξ, γ) plane. Each side separatrix (SS) joins two unstable stationary points at $\xi=0$ or $\xi=1$. The branches of these separatrices surround a single stable stationary point at $\gamma=\pi/2$ or $\gamma=0$ $(0<\xi<1)$. One can verify that if, for example, between the "left" separatrices there is a point on the line $\gamma=0$, then the abscissa of the point of intersection of the separatrix with the line $\gamma=0$ is double the abscissa of the stationary point ξ_*; it is obvious that $\xi<\frac{1}{2}$. A similar property is satisfied by the right separatrix: here it is necessary for the stationary point surrounded by its branches to be in the right half of the rectangle. The separatrix emerging from $\xi=0$ cannot intersect the line $\xi=1$, and conversely.

The branches of the *CS* join the two unstable stationary points (3.4), corresponding to even or odd values of n, and surrounding the stable stationary point. The CS cannot intersect the lines $\xi=0$ or $\xi=1$. Inside the domains surrounded by the SS or CS a modulation regime of the first type exists, and outside these domains, a regime of the second type.

Thus four qualitatively different types of AP-portrait are possible, governed by conditions 1–6, and they are shown in Fig. 2.

1. Conditions 1 and 2 are satisfied. There are stable stationary points at $\gamma=n\pi/2$ (3.4) for even and odd n, in the left section $(\xi<\frac{1}{2})$ and right section $(\xi>\frac{1}{2})$ of the rectangle, i.e. three stable normal modes exist (and two trivial unstable ones $u_k=0$, $k=1, 2$). Each of the stationary points is "captured" by the corresponding SS; there are no CS (Fig. 2a).

2. Only one of conditions 1 and 2 is satisfied. There is a stable stationary point (3.4) only for an odd or even n, and only one SS (on the left if condition 1 is satisfied, and on the right if condition 2 is satisfied); there are no CS (Fig. 2b). Of the normal modes, apart from a single $u_k=0$, $k=1$ or $k=2$ mode, stable modes also exist that are either rectilinear (if condition 4 is satisfied), or elliptic (when condition 3 is satisfied).

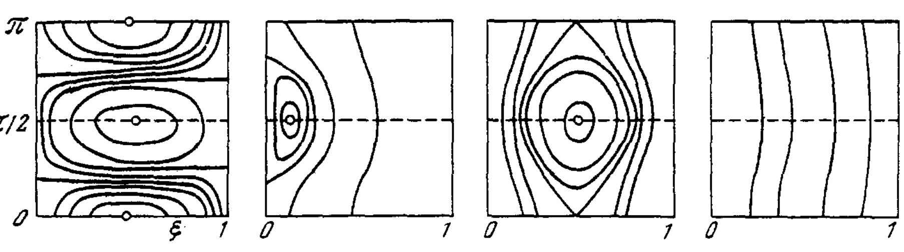


FIG. 2.

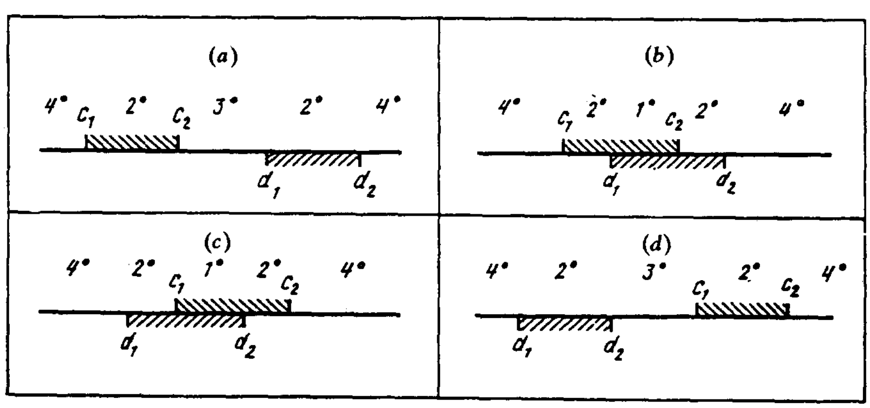


FIG. 3.

3. Neither condition 1 nor 2 is satisfied, but condition 3 is satisfied. The stable and unstable stationary points (3.4) alternate (with the point for odd n being stable if condition 5 is satisfied). There is a CS, but no SS (Fig. 2c). Three normal modes exist, where either the rectilinear one is stable (when condition 6 is satisfied), or the elliptic one (when condition 5 holds).

4. Conditions 1–3 are not satisfied. There are no stationary points (normal modes) or separatrices. All oscillatory modes are of modulation type 2, with the modulation being relatively small compared with cases 1–3 (Fig. 2d).

In cases 1–3 one can distinguish subcases. In case 1 there are two subcases distinguished by the position of the left stationary point: on the $\gamma = 0$ line or on $\gamma = \pi/2$. Similarly, in case 3 the stationary point can be stable at $\gamma = 0$ or at $\gamma = \pi/2$. Four subcases are possible for case 2: a left or right separatrix, and a stationary point at $\gamma = 0$ or $\gamma = \pi/2$. The corresponding AP-portraits can be obtained from those shown in Fig. 2.

4. THE INFLUENCE OF FREQUENCY "SEPARATION" ON SYSTEM BEHAVIOUR

We introduce the parameters

$$\alpha_1 = b_{11}/b_{12}, \quad \alpha_2 = b_{22}/b_{12}, \quad \sigma^0 = 4\sigma/(b_{12}E) \tag{4.1}$$

Then conditions 1–6 can be represented in the form

$$\begin{aligned}
&(1)\ 3\alpha_2 - 3 \leqslant \sigma^0 \leqslant 3\alpha_2 - 1 && \\
&(2)\ -3\alpha_1 + 1 \leqslant \sigma^0 \leqslant -3\alpha_1 + 3 && \\
&(3)\ -3\alpha_1 + 1 \leqslant \sigma^0 \leqslant 3\alpha_2 - 1 && \text{for} \quad \alpha_1 + \alpha_2 > 2/3 \\
&\quad\ 3\alpha_2 - 1 \leqslant \sigma^0 \leqslant -3\alpha_1 + 1 && \text{for} \quad \alpha_1 + \alpha_2 < 2/3 \\
&(4)\ 3\alpha_2 - 3 \leqslant \sigma^0 \leqslant -3\alpha_1 + 3 && \text{for} \quad \alpha_1 + \alpha_2 < 2 \\
&\quad\ -3\alpha_1 + 3 \leqslant \sigma^0 \leqslant 3\alpha_2 - 3 && \text{for} \quad \alpha_1 + \alpha_2 > 2 \\
&(5)\ \alpha_1 + \alpha_2 > 2/3 && \\
&(6)\ \alpha_1 + \alpha_2 < 2 &&
\end{aligned} \tag{4.2}$$

Unlike σ and E, the dimensionless frequency separation parameter σ^0 does not depend on the choice of ε and can be written in the following form

$$\sigma^0 = 4\sigma_* b_{12}^{-1}[u_1^2(0) + u_2^2(0)]^{-1}, \quad (\sigma_* = \varepsilon^2\sigma = \omega_2^2 - \omega_1^2) \tag{4.3}$$

As can be seen from (4.2), the type of AP-portrait is determined from the relative positions of the points

$$c_1 = 3\alpha_2 - 3, \quad c_2 = 3\alpha_2 - 1, \quad d_1 = -3\alpha_1 + 1, \quad d_2 = -3\alpha_1 + 3 \tag{4.4}$$

and the quantity σ^0. Four possible positionings of the intervals (c_1, c_2) and (d_1, d_2) are shown in Fig. 3 ($c_2 < d_1$, $c_1 < d_1 < c_2$, $c_1 < d_2 < c_2$, $d_2 < c_2$). The type of AP-portrait (easily determined from (4.2)) is shown above the intervals. In case (a) the interval (c_2, d_1) contains the stable stationary point at $\gamma = 0(\pi)$ (i.e. the rectilinear normal mode) is stable, and the unstable one is at $\gamma = \pi/2$ $(3\pi/2)$ (i.e. elliptic). In case (d) these points (and normal oscillations) "exchange" stability.

Figure 3 graphically demonstrates the influence of the parameter σ^0 on the system behaviour. If σ^0 lies in the interval

$$\delta_1 < \sigma^0 < \delta_2, \quad \delta_1 = \min(c_1, d_1), \quad \delta_2 = \max(c_2, d_2) \tag{4.5}$$

then we have AP-portraits of types 1–3 with stationary points and pronounced modulation of the amplitude and phase (energy exchange). If σ^0 lies outside this interval, an AP-portrait of type 4 is indicated with relatively small modulation. Thus condition (4.5) allows one to specify the concept of small frequency separation. The minimum width of interval (4.5) is 2. The centre of the interval is the point

$$\alpha_* = \tfrac{3}{2}(\alpha_2 - \alpha_1) = \tfrac{3}{2}(b_{22} - b_{11}) / b_{12}$$

In the case when $b_{11} = b_{22}$ we have $\alpha_1 = \alpha_2$: $\alpha_* = 0$, i.e. interval (4.5) is symmetric about the origin.

If $\alpha_1 \neq \alpha_2$ the interval is displaced relative to the origin and for sufficiently large $|\alpha_2 - \alpha_1|$ (or $|b_{22} - b_{11}|$) the point $\sigma^0 = 0$ can turn out to lie outside the interval. One must also take into account that the sign of σ^0 is governed by the sign of b_{12} (one can always put $\sigma_* > 0$, i.e. $\omega_2 > \omega_1$). The sign-constancy condition on σ^0 singles out either the positive or negative part of interval (4.5) (if it exists). Two conclusions follow from this:

1. it is not necessary for the larger modulation to correspond to the smaller value of σ_0: combinations of the coefficients b_{ij} are possible with AP-portraits of types 1–3 for intervals with σ^0 far from the point 0;

2. for certain combinations of b_{ij} only type 4 AP-portraits are possible, irrespective of the energy and frequency separation.

In the above analysis there is a natural separation of the influence on the energy exchange of the oscillation energy and the ratio of the initial amplitudes of the two modes. The quantity E acts on σ^0 according to (4.3) (increasing E being equivalent to decreasing σ), and together with σ_* it therefore determines the type of AP-portrait. The initial amplitude ratio ξ_0 determines the phase trajectory in a given AP-portrait.

Consider the special case when $b_{11} = b_{22} = 0$, $b_{12} \neq 0$. Then $\alpha_1 = \alpha_2 = 0$, $c_1 = -3$, $c_2 = -1$, $d_1 = 1$, $d_2 = 3$, i.e. we have Fig. 3, case (a). Condition 5 is not satisfied, while condition 6 is satisfied. For $-3 < \sigma^0 \leqslant -1$ we have a type 2 AP-portrait with left separatrix and stable stationary point at $\gamma = 0(\pi)$, i.e. with rectilinear normal oscillations. For $-1 < \sigma^0 < 1$ the AP-portrait is of type 3 and has a stable stationary point at $\gamma = 0(\pi)$ and an unstable one at $\gamma = \pi/2$ $(3\pi/2)$, i.e. with stable rectilinear normal modes and an unstable elliptic mode. When $1 < \sigma^0 < 3$ the AP-portrait is of type 2 with a right separatrix and stable rectilinear normal modes. Finally, for $\sigma^0 < -3$ and $\sigma^0 > 3$ the AP-portrait is of type 4.

In conclusion we note that numerical integrations of Eq. (1.1), performed for the purpose of estimating the accuracy of the solution obtained by the multiscale method, demonstrated almost complete agreement between the analytic and numerical solutions in all the cases considered with an arbitrary choice of $\varepsilon \leqslant 0.1$ and amplitudes of up to 0.5 (the error in determining the amplitude being of the order of 0.1%). But when ε was increased beyond 0.1, the error increased rapidly. For example, when $\varepsilon = 0.15$ the error in the amplitude computation reached 30%.

REFERENCES

1. NAYFEH A. H. and MOOK D. T., *Nonlinear Oscillations*. John Wiley, New York, 1979.
2. ABDEL-GHAFFAR A. M. and RUBIN L. I., Non-linear free vibrations of suspension bridges: theory and applications. *J. Engng Mech. Am. Soc. Civil Engrs* **109**, 313–345, 1983.
3. ROSSIKHIN Yu. A. and SHITIKOVA M. V., Non-linear free spatial oscillations of suspended composite systems. *Prikl. Mat. Mekh.* **54**, 1003–1011, 1990.

4. ROSSIKHIN Yu. A. and SHITIKOVA M. V., The influence of initial conditions on the nature of oscillatory processes in a suspended composite system. *Izv. Akad. Nauk SSSR, MTT* 3, 143–154, 1991.
5. AKSETYAN G. K. and POPOV N. A., Non-linear proper oscillations of a vibration-damped system with two degrees of freedom. *Izv. Vuz., Stroitel'stvo i Arkhitektura* 8, 53–57, 1978.
6. CHEN J. C. and BABCOCK C. D., Non-linear vibration of cylindrical shells. *AIAA Jnl* **13**, 868–876, 1975.
7. KUBENKO V. D., KOVAL'CHUK P. S. and KRASNOPOL'SKAYA T. S., *Non-linear Interaction of the Bending Oscillation Modes of Cylindrical Shells*. Nauk. Dumka, Kiev, 1984.
8. ROSENBERG R. M. and HSU C. S., On the geometrization of normal vibrations of non-linear systems having many degrees of freedom. In *Proceeding of the International Symposium on Non-linear Oscillations*, Vol. 1, *Analytical Methods of Non-linear Oscillation Theory*, pp. 380–415. Izd-vo Akad. Nauk IkrSSR, Kiev, 1963.

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